Holonomy on Stiefel Bundles over Grassmannian manifolds

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Abstract. Consider Stiefel bundles over Grassmann manifolds $U(n) \to U(n+m)/U(m) \xrightarrow{\pi} G_{n,m}$. Let S be a complete totally geodesic surface in the base space and γ be a piecewise smooth, simple closed curve on S. Then the holonomy displacement along γ is given by

$$V(\gamma) = e^{\lambda\, A(\gamma) i}$$

where $A(\gamma)$ is the area of the region on the surface S surrounded by γ; $\lambda = \frac{1}{2}$ or 0 depending on whether S is a complex submanifold or not.

In the process, we also characterize complete totally geodesic 2-dimensional submanifolds in Grassmanian manifolds $G_{n,m}$.

0. Introduction

Consider the Hopf fibration $S^1 \to S^3 \to S^2$. Let γ be a simple closed curve on S^2. Pick a point in S^3 over $\gamma(0)$, and take the unique horizontal lift $\widetilde{\gamma}$ of γ. Since $\gamma(1) = \gamma(0)$, $\widetilde{\gamma}(1)$ lies in the same fiber as $\widetilde{\gamma}(0)$ does. We are interested in understanding the difference between $\widetilde{\gamma}(0)$ and $\widetilde{\gamma}(1)$. The following equality was already known [4]:

$$V(\gamma) = e^{\frac{1}{2}A(\gamma) i},$$

where $V(\gamma)$ is the holonomy displacement along γ, and $A(\gamma)$ is the area of the region surrounded by γ.

In this paper, we shall generalize this fact to the following higher dimensional Stiefel bundle over Grassmannian manifold:

$$U(n) \to U(n+m)/U(m) \xrightarrow{\pi} G_{n,m},$$

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where $G_{n,m} = U(n+m)/\big(U(n)\times U(m)\big)$. The main results are stated as follows:

Let S be a complete totally geodesic surface in the base space. Let γ be a piecewise smooth, simple closed curve on S parametrized by $0 \le t \le 1$, and $\widetilde{\gamma}$ its horizontal lift on the bundle $U(n) \to \pi^{-1}(S) \xrightarrow{\pi} S$, which is immersed in $U(n) \to U(n+m)/U(m) \xrightarrow{\pi} G_{n,m}$. Then

$$\widetilde{\gamma}(1) = e^{\frac{1}{2}A(\gamma)i}\cdot\widetilde{\gamma}(0) \qquad \text{or} \qquad \widetilde{\gamma}(1) = \widetilde{\gamma}(0),$$

where $A(\gamma)$ is the area of the region on the surface S surrounded by γ, depending on whether S has a complex submanifold or not. See Theorem 2.6.

We also characterize complete totally geodesic 2-dimensional submanifolds in Grassmanian manifolds $G_{n,m}$.

1. The bundle $U(1) \longrightarrow U(2)/U(1) \longrightarrow G_{1,1}$

First we study the case of $n = m = 1$ for the general principal bundle

$$U(n) \longrightarrow U(n+m)/U(m) \longrightarrow G_{n,m}\,.$$

We use $SU(2)$ rather than $U(2)$. Thus, our bundle is

$$U(1) \longrightarrow SU(2) \longrightarrow SU(2)/U(1).$$

Of course,

$$S^3 \cong SU(2) = \{A \in \mathrm{GL}(2,\mathbb{C}) \ : \ AA^* = I \text{ and } \det(A) = 1\}.$$

From now on, we shall use the convention of $\mathfrak{gl}(k,\mathbb{C}) \subset \mathfrak{gl}(2k,\mathbb{R})$ by

$$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \longrightarrow \begin{bmatrix} x_{11}+iy_{11} & x_{12}+iy_{12} \\ x_{21}+iy_{21} & x_{22}+iy_{22} \end{bmatrix} \longrightarrow \begin{bmatrix} x_{11} & -y_{11} & x_{12} & -y_{12} \\ y_{11} & x_{11} & y_{12} & x_{12} \\ x_{21} & -y_{21} & x_{22} & -y_{22} \\ y_{21} & x_{21} & y_{22} & x_{22} \end{bmatrix}.$$

The group $SU(2)$ has the following natural representation into $\mathrm{GL}(4,\mathbb{R})$:

$$w = \begin{bmatrix} w_1 & w_2 & -w_3 & -w_4 \\ -w_2 & w_1 & w_4 & -w_3 \\ w_3 & -w_4 & w_1 & -w_2 \\ w_4 & w_3 & w_2 & w_1 \end{bmatrix}$$

with the condition $w_1^2 + w_2^2 + w_3^2 + w_4^2 = 1$. In fact, the map

$$w_1 + w_2 i + w_3 j + w_4 k \longmapsto w$$

is a monomorphism from the unit quaternions into $\mathrm{GL}(4,\mathbb{R})$. The circle group

$$S^1 = \left\{ \begin{bmatrix} e^{-iz} & 0 \\ 0 & e^{iz} \end{bmatrix} \ : \ 0 \le z \le 2\pi \right\}$$

is a subgroup of $SU(2)$, and acts on $SU(2)$ as right translations, freely with quotient $\mathbb{C}P^1 = S^2$, the 2-sphere, giving rise to the fibration

$$S^1 \longrightarrow SU(2) \longrightarrow \mathbb{C}P^1.$$

Let $\widetilde{w}$ be the "i-conjugate" of w (replace w_2 by $-w_2$). That is,

$$\widetilde{w} = \begin{bmatrix} w_1 & -w_2 & -w_3 & -w_4 \\ w_2 & w_1 & w_4 & -w_3 \\ w_3 & -w_4 & w_1 & w_2 \\ w_4 & w_3 & -w_2 & w_1 \end{bmatrix}.$$

Then,

$$w\widetilde{w} = \begin{bmatrix} w_1^2+w_2^2-w_3^2-w_4^2 & 0 & -2(w_1w_3+w_2w_4) & 2w_2w_3-2w_1w_4 \\ 0 & w_1^2+w_2^2-w_3^2-w_4^2 & -2w_2w_3+2w_1w_4 & -2(w_1w_3+w_2w_4) \\ 2(w_1w_3+w_2w_4) & 2w_2w_3-2w_1w_4 & w_1^2+w_2^2-w_3^2-w_4^2 & 0 \\ -2w_2w_3+2w_1w_4 & 2(w_1w_3+w_2w_4) & 0 & w_1^2+w_2^2-w_3^2-w_4^2 \end{bmatrix}$$

and

$$(w_1^2+w_2^2-w_3^2-w_4^2)^2 + (2w_1w_3+2w_2w_4)^2 + (-2w_2w_3+2w_1w_4)^2 = 1.$$

Clearly, $\mathbb{C}P^1$ can be identified with the following

$$\mathbb{C}P^1 = \left\{ \begin{bmatrix} x & 0 & -y & -z \\ 0 & x & z & -y \\ y & -z & x & 0 \\ z & y & 0 & x \end{bmatrix} : \; x^2+y^2+z^2 = 1 \right\}.$$

Therefore, the map

$$p : SU(2) \longrightarrow \mathbb{C}P^1$$

defined by

$$p(w) = w\widetilde{w}$$

has the following properties

$$\begin{aligned} p(wv) &= wp(v)\widetilde{w} \quad \text{for all } w, v \in SU(2) \\ p(wv) &= p(w) \quad \text{if and only if} \quad v \in S^1. \end{aligned}$$

This shows that the map p is, indeed, the orbit map of the principal bundle $S^1 \longrightarrow SU(2) \longrightarrow \mathbb{C}P^1$.

The Lie group $SU(2)$ will have a left-invariant Riemannian metric given by the following orthonormal basis on the Lie algebra $\mathfrak{su}(2)$

$$e_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Notice that e_1 and e_2 correspond to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ i \end{bmatrix}$ in $\mathfrak{gl}(2, \mathbb{C})$ and $[e_1, e_2] = 2e_3$. In order to understand the projection map better, consider

the subset of $SU(2)$:

$$T = \left\{ \begin{bmatrix} \cos x & -(\sin x)e^{-iy} \\ (\sin x)e^{iy} & \cos x \end{bmatrix} \;:\; 0 \le x \le \pi,\; 0 \le y \le 2\pi \right\}$$
$$= \left\{ \begin{bmatrix} \cos x & 0 & -(\sin x)(\cos y) & -(\sin x)(\sin y) \\ 0 & \cos x & (\sin x)(\sin y) & -(\sin x)(\cos y) \\ (\sin x)(\cos y) & -(\sin x)(\sin y) & \cos x & 0 \\ (\sin x)(\sin y) & (\sin x)(\cos y) & 0 & \cos x \end{bmatrix} \right\}$$

which is the exponential image of

$$\mathfrak{m} = \left\{ \begin{bmatrix} 0 & -\bar{\xi}^t \\ \xi & 0 \end{bmatrix} \;:\; \xi \in \mathbb{C} \right\}.$$

The map p restricted to T is just the squaring map; that is,

$$p(w) = w^2, \quad w \in T.$$

Theorem 1.1 ([4])**.** *Let $S^1 \to SU(2) \to \mathbb{C}P^1$ be the natural fibration. Let γ be a piecewise smooth, simple closed curve on $\mathbb{C}P^1$. Then the holonomy displacement along γ is given by*

$$V(\gamma) = e^{\frac{1}{2}A(\gamma)i} \;\in S^1$$

where $A(\gamma)$ is the area of the region on $\mathbb{C}P^1$ enclosed by γ.

Proof. Let $\gamma(t)$ be a closed loop on $\mathbb{C}P^1$ with $\gamma(0) = p(I_4)$. Therefore,

$$\gamma(t) = \begin{bmatrix} \cos 2x(t) & 0 & -\sin 2x(t)\cos y(t) & -\sin 2x(t)\sin y(t) \\ 0 & \cos 2x(t) & \sin 2x(t)\sin y(t) & -\sin 2x(t)\cos y(t) \\ \sin 2x(t)\cos y(t) & -\sin 2x(t)\sin y(t) & \cos 2x(t) & 0 \\ \sin 2x(t)\sin y(t) & \sin 2x(t)\cos y(t) & 0 & \cos 2x(t) \end{bmatrix}$$

Let

$$\widetilde{\gamma}(t) = \begin{bmatrix} \cos x(t) & 0 & -\sin x(t)\cos y(t) & -\sin x(t)\sin y(t) \\ 0 & \cos x(t) & \sin x(t)\sin y(t) & -\sin x(t)\cos y(t) \\ \sin x(t)\cos y(t) & -\sin x(t)\sin y(t) & \cos x(t) & 0 \\ \sin x(t)\sin y(t) & \sin x(t)\cos y(t) & 0 & \cos x(t) \end{bmatrix}$$

with $0 \le x(t) \le \pi/2$ so that $p(\widetilde{\gamma}(t)) = \gamma(t)$ ($\widetilde{\gamma}$ is a lift of γ), and let

$$\omega(t) = \begin{bmatrix} \cos z(t) & -\sin z(t) & 0 & 0 \\ \sin z(t) & \cos z(t) & 0 & 0 \\ 0 & 0 & \cos z(t) & \sin z(t) \\ 0 & 0 & -\sin z(t) & \cos z(t) \end{bmatrix}.$$

Put

$$\eta(t) = \widetilde{\gamma}(t) \cdot \omega(t).$$

Then still $p(\eta(t)) = \gamma(t)$, and η is another lift of γ. We wish η to be the horizontal lift of γ. That is, we want $\eta'(t)$ to be orthogonal to the fiber at $\eta(t)$.

The condition is that $\langle \eta'(t), (\ell_{\eta(t)})_*(e_3)\rangle = 0$, or equivalently, $\langle (\ell_{\eta(t)^{-1}})_*\eta'(t), e_3\rangle = 0$. That is,

$$\eta(t)^{-1} \cdot \eta'(t) = \alpha_1 e_1 + \alpha_2 e_2$$

for some $\alpha_1, \alpha_2 \in \mathbb{R}$. From this, we get the following equation:

$$z'(t) = \sin^2 x(t) y'(t). \tag{1–1}$$

Since any piecewise smooth curve can be approximated by a sequence of piecewise linear curves which are sums of boundaries of rectangular regions, it will be enough to prove the statement for a particular type of curves as follows [1]: Suppose we are given a rectangular region in the xy-plane

$$p \le x \le p + a$$
$$q \le y \le q + b.$$

Consider the image R of this rectangle in $\mathbb{C}P^1$ by the map

$$(x, y) \mapsto \mathbf{r}(x, y) = (\cos 2x, (\sin 2x)(\cos y), (\sin 2x)(\sin y)).$$

The area of R can be calculated as follows:

$$\mathbf{r}_x \times \mathbf{r}_y = ((\cos 2x)(2\sin 2x), (2\sin^2 2x)(\cos y), (2\sin^2 2x)(\sin y)).$$

Now

$$||\mathbf{r}_x \times \mathbf{r}_y|| = 2\sin 2x, \quad \text{(because } 0 \le x \le \pi/2).$$

Thus, the area is

$$\int_q^{q+b}\int_p^{p+a} 2\sin 2x \; dxdy = 2b(\sin^2(p+a) - \sin^2(p)).$$

On the other hand, the change of $z(t)$ along the boundary of this region can be calculated using condition (1–1). Label the four vertices by $A(p, q)$, $B(p + a, q)$, $C(p + a, q + b)$, and $D(p, q + b)$. AB can be parametrized by $x(t) = p + at$, $y(t) = q$, $t \in [0, 1]$ so that $y'(t) = 0$. For BC, $x(t) = p + a$, $y(t) = q + bt$, $t \in [0, 1]$. Then

$$z(1) - z(0) = \int_0^1 z'(t)dt = \int \sin^2(p + a)bdt = b \cdot \sin^2(p + a).$$

Similarly, $z(t)$ does not change along CD, but on DA, $x(t) = p$, $y(t) = q + b - bt$, $t \in [0, 1]$. So

$$z(1) - z(0) = \int_0^1 z'(t)dt = \int \sin^2(p)(-b)dt = -b \cdot \sin^2(p).$$

Thus the total vertical change of z-values, $z(1) - z(0)$, along the perimeter of this rectangle is

$$b \cdot (\sin^2(p + a) - \sin^2(p))$$

which is $\frac{1}{2}$ times the area. Hence we get the conclusion. $\square$

2. The bundle $U(n) \longrightarrow U(n+m)/U(m) \longrightarrow G_{n,m}$

To deal with the bundle

$$U(n) \to U(n+m)/U(m) \to G_{n,m}\,,$$

we investigate the bundle

$$U(n) \times U(m) \to U(n+m) \to G_{n,m}\,.$$

The Lie algebra of $U(n+m)$ is $\mathfrak{u}(n+m)$, the skew-Hermitian matrices, and has the following canonical decomposition:

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m},$$

where

$$\mathfrak{h} = \mathfrak{u}(n) + \mathfrak{u}(m) = \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \ : \ A \in \mathfrak{u}(n),\ B \in \mathfrak{u}(m) \right\}$$

and

$$\mathfrak{m} = \left\{ \hat{X} := \begin{bmatrix} 0 & -X^* \\ X & 0 \end{bmatrix} \ : \ X \in M_{m,n}(\mathbb{C}) \right\}.$$

Define an Hermitian inner product $h : \mathbb{C}^m \to \mathbb{C}$ by

$$h(v, w) = v^* \, w,$$

where v and w are regarded as column vectors.

Lemma 2.1. *Let*

$$X = \left(a_k^r + i b_k^r\right), Y = \left(c_k^r + i d_k^r\right) \in M_{m,n}(\mathbb{C})$$

for $r = 1, \cdots, m,$ and $k = 1, \cdots, n$. Suppose that for their induced $\hat{X}, \hat{Y} \in \mathfrak{m}$,

$$[[\hat{X}, \hat{Y}], \hat{X}] = \hat{Z} \in \mathfrak{m}$$

for some $Z = \left(\alpha_k^r\right) \in M_{m,n}(\mathbb{C})$ for $r = 1, \cdots, m,$ and $k = 1, \cdots, n$. Then we have

$$\alpha_k^r = \sum_{j=1}^{n} (a_j^r + i b_j^r)\big(-2h(Y_j, X_k) + h(X_j, Y_k)\big) + \sum_{j=1}^{n} (c_j^r + i d_j^r)\, h(X_j, X_k)$$

where X_k and Y_k are k-column vectors of X and Y for $k = 1, \cdots, n$.

Proof. It is easily obtained from

$$[[\hat{X}, \hat{Y}], \hat{X}] = \hat{X}(2\hat{Y}\hat{X} - \hat{X}\hat{Y}) - \hat{Y}\hat{X}\hat{X}.$$

$\square$

Theorem 2.2. *Let $U(n) \times U(m) \to U(n+m) \to G_{n,m}(\mathbb{C}), n \le m,$ be the natural fibration. Assume a 2-dimensional subspace $\mathfrak{m}' = Span_{\mathbb{R}}\{\hat{X}, \hat{Y}\}$ of $\mathfrak{m} \subset \mathfrak{u}(n+m)$ satisfies*

$$X^* X = \lambda I_n, \quad X^* Y = \mu I_n, \qquad \lambda \in \mathbb{R} - \{0\}, \mu \in \mathbb{C} \tag{2–1}$$

for $X, Y \in M_{m,n}(\mathbb{C})$. Then $\mathfrak{m}'$ gives rise to a complete totally geodesic surface S in $G_{n,m}(\mathbb{C})$ if and only if either

(1) $[\hat{X}, \hat{Y}] \in \mathfrak{u}(m)$ *and* $Y^*Y = \eta I_n$ *for some* $\eta \in \mathbb{R}$ *in case of* $\operatorname{Im}\mu = 0$,
(2) $\mathfrak{m}'$ *is* J*-invariant (i.e., has a complex structure) in case of* $\operatorname{Im}\mu \neq 0$.

Proof. Assume that $\mathfrak{m}'$ gives rise to a complete totally geodesic surface S in $G_{n,m}(\mathbb{C})$. If $\mathrm{Im}\mu = 0$, then $-X^*Y + Y^*X = -X^*Y + (X^*Y)^* = -2i\mathrm{Im}\mu\, I_n = O_n$, so

$$[\hat{X}, \hat{Y}] = \begin{bmatrix} O_n & 0 \\ 0 & -XY^* + YX^* \end{bmatrix} \in \mathfrak{u}(m) \subset \mathfrak{u}(n+m).$$

Let $M = -XY^* + YX^*$. Then

$$[\hat{X}, \hat{Y}] = \begin{bmatrix} O_n & 0 \\ 0 & M \end{bmatrix}$$

and $[[\hat{Y}, \hat{X}], \hat{Y}] = -\widehat{MY} \in \mathfrak{m}'$ from the hypothesis of the condition of totally geodesic. Note that

$$-MY = XY^*Y - YX^*Y = XY^*Y - Y\mu I_n = XY^*Y - (\mathrm{Re}u)Y.$$

Thus $XY^*Y = aX + bY$ for some $a, b \in \mathbb{R}$. Then $\lambda Y^*Y = X^*(XY^*Y) = X^*(aX + bY) = (a\lambda + b\mathrm{Re}\mu)I_n$ and so

$$Y^*Y = \tfrac{a\lambda + b\mathrm{Re}\mu}{\lambda} I_n, \quad \tfrac{a\lambda + b\mathrm{Re}\mu}{\lambda} \in \mathbb{R}.$$

Suppose that $\mathrm{Im}\mu \neq 0$. Let $e_k \in \mathbb{C}^m$, $k = 1, \cdots, m$, be an elementary vector which has all components 0 except for the k-component with 1. Then

$$h(X_k, Y_j) = h(Xe_k, Ye_j) = e_k^*(X^*Y)e_j.$$

Then the condition (2–1) is equivalent to

$$h(X_k, Y_k) = \mu, \quad h(X_k, X_k) = \lambda, \quad h(X_k, X_j) = 0, \quad h(X_k, Y_j) = 0$$

for $k \neq j$ in $\{1, \cdots, n\}$. From $h(X_k, Y_k) = \mu$, we obtain

$$-2h(Y_k, X_k) + h(X_k, Y_k) = -\mathrm{Re}\mu + 3i\mathrm{Im}\mu$$

Thus Lemma 2.1 says that

$$\begin{aligned} [[\hat{X}, \hat{Y}], \hat{X}] &= (-\mathrm{Re}\mu + 3i\mathrm{Im}\mu)\hat{X} + \lambda\hat{Y} \\ &= 3\mathrm{Im}\mu(i\hat{X}) + (-\mathrm{Re}\mu\hat{X} + \lambda\hat{Y}). \end{aligned}$$

From the hypothesis of the condition of totally geodesic, $[[\hat{X}, \hat{Y}], \hat{X}] = a\hat{X} + b\hat{Y}$ for some $a, b \in \mathbb{R}$. Since $\mathrm{Im}\mu \neq 0$, $i\hat{X}$ will lie in $\mathrm{Span}_{\mathbb{R}}\{\hat{X}, \hat{Y}\} = \mathfrak{m}'$, which implies that $\mathfrak{m}'$ will be J-invariant.

Conversely, assume the necessary part holds. If the condition (1) holds, then $[[\hat{X}, \hat{Y}], \hat{X}] = \widehat{MX}$ and $[[\hat{Y}, \hat{X}], \hat{Y}] = -\widehat{MY}$, where $M = -XY^* + YX^*$. It suffices to show that $[[\hat{X}, \hat{Y}], \hat{X}] \in \mathfrak{m}'$ and $[[\hat{Y}, \hat{X}], \hat{Y}] \in \mathfrak{m}'$. Since

$$MX = -XY^*X + YX^*X = -X\bar{\mu}I_n + Y\lambda I_n = -\mathrm{Re}uX + \lambda Y,$$

we get $[[\hat{X},\hat{Y}],\hat{X}] \in \mathfrak{m}'$. We also get $[[\hat{Y},\hat{X}],\hat{Y}] \in \mathfrak{m}'$ since

$$-MY = XY^*Y - YX^*Y = X\eta I_n - Y\mu I_n = \eta X - \mathrm{Re}uY.$$

If the condition (2) holds, then $\mathfrak{m}' = \mathrm{Span}_{\mathbb{R}}\{\hat{X}, \widehat{iX}\}$, and

$$[\hat{X},\widehat{iX}] = \begin{bmatrix} -2i\lambda I_n & 0 \\ 0 & 2iXX^* \end{bmatrix}.$$

It suffices to show that $[[\hat{X},\widehat{iX}],\hat{X}] \in \mathfrak{m}'$ and $[[\hat{X},\widehat{iX}],\widehat{iX}] \in \mathfrak{m}'$. Since $[\hat{X},\widehat{iX}] \in \mathfrak{u}(n+m)$, XX^* will be an element in $\mathfrak{u}(m)$, so O_m. Thus, $[\hat{X},\widehat{iX}] = -2i\lambda \begin{bmatrix} I_n & 0 \\ 0 & O_m \end{bmatrix}$, and so $[[\hat{X},\widehat{iX}],\hat{X}] = 2\lambda\widehat{iX}$ and $[[\hat{X},\widehat{iX}],\widehat{iX}] = -2\lambda\hat{X}$. Hence we get the conclusion. □

Remark 2.3. The condition of X in Theorem 2.2 says $X : \mathbb{C}^n \to \mathbb{C}^m$ is a conformal one-one linear map. In view of $\hat{X} \in \mathfrak{u}(n+m) \subset \mathrm{End}(\mathbb{C}^{n+m})$, $\hat{X}$ sends the subspace $\mathbb{C}^n$ to its orthogonal subspace $\mathbb{C}^m$ conformally. And the condition of the relation between X and Y says that

$$h_{\mathbb{C}^m}(Xv, Yw) = \mu\, h_{\mathbb{C}^n}(v,w) \qquad \text{for } v, w \in \mathbb{C}^n,$$

where $h_{\mathbb{C}^k}$ is an Hermitian on $\mathbb{C}^k$, $k = 1, 2, \cdots$, given by

$$h_{\mathbb{C}^k}(u_1,u_2) = u_1^* u_2 \quad \text{for } u_1, u_2 \in \mathbb{C}^k.$$

When $n = 1$, the condition (2–1) is satisfied automatically for any two vectors in $\mathbb{C}^m$ by identifying $M_{m,1}(\mathbb{C})$ with $\mathbb{C}^m$. So we get

Corollary 2.4. *A 2-dimensional subspace $\mathfrak{m}'$ of $\mathfrak{m} \subset \mathfrak{u}(m+1)$ gives rise to a complete totally geodesic submanifold of $\mathbb{C}P^m$ if and only if either*

(1) *$\mathfrak{m}'$ is J-invariant (i.e., has a complex structure), or*
(2) *$\mathfrak{m}'$ has tangent vectors $\hat{v}$ and $\hat{w}$ such that $\mathrm{Im}h_{\mathbb{C}^m}(v,w) = 0$.*

We return to the bundle $U(n) \to U(n+m)/U(m) \xrightarrow{\pi} G_{n,m}$. Any subset $A \subset G_{n,m}$ induces a bundle $U(n) \to \pi^{-1}(A) \to A$, which is immersed in the original bundle and diffeomorphic to the pullback bundle with respect the inclusion of A into $G_{n,m}$. In fact, in the bundle $U(n)\times U(m) \to U(n+m) \xrightarrow{p} G_{n,m}$, the induced distribution in $p^{-1}(A)$ from $\mathfrak{u}$(m) in $U(n+m)$ is integrable, so this induces the bundle $U(n) \to \pi^{-1}(A) \to A$.

Theorem 2.5. *Assume the same condition for a complete totally geodesic surface S of Theorem 2.2. Then, in the bundle $U(n) \to \pi^{-1}(S) \to S$, which is immersed in the original bundle $U(n) \to U(n+m)/U(m) \xrightarrow{\pi} G_{n,m}$, either*

(1) *it is flat in case of $\mathrm{Im}\,\mu = 0$, or*
(2) *there exist a subbundle of rank 1, which is isomorphic to the Hopf bundle $S^1 \to S^3 \to S^2$ in case of $\mathrm{Im}\,\mu \neq 0$.*

Proof. Assume that $\mathrm{Im}\mu = 0$. Consider the bundle $U(n) \times U(m) \to U(n+m) \xrightarrow{p} G_{n,m}$. Then S induces a bundle $U(n) \times U(m) \to p^{-1}(S) \to S$. Totally geodesic condition says that the distribution induced from $\mathrm{Span}_{\mathbb{R}}\{X, Y, [X, Y]\}$ is integrable. Since $[X, Y]$ is contained in the Lie algebra $\mathfrak{u}(m)$ of $U(m)$, (1) is obtained.

Assume that $\mathrm{Im}\mu \neq 0$. Consider the following three elements in $\mathfrak{su}(1+1)$:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}.$$

Since $\mathfrak{m}'$ is J-invariant, there is a Lie algbra monomorphism $f : \mathfrak{s}u(1+1) \to \mathfrak{u}(n+m)$, given by

$$f(aA + bB + cC) = a\hat{X} + b\widehat{iX} + cK,$$

where $K = \begin{bmatrix} -\lambda I_n & 0 \\ 0 & O_m \end{bmatrix} \in \mathfrak{u}(n)$. In fact,

$$[A, B] = 2C, \quad [C, A] = 2B, \quad [C, B] = -2A$$

and

$$[\hat{X}, \widehat{iX}] = 2K, \quad [K, \hat{X}] = 2\widehat{iX}, \quad [K, \widehat{iX}] = -2\hat{X}.$$

Thus f will induce a Lie group monomorphism $\tilde{f} : SU(1+1) \to U(n+m)$ with $\tilde{f}\Big(S\big(U(1) \times U(1)\big)\Big) \subset U(n) \times U(m)$ since $SU(2)$ is simply connected and $S\big(U(1) \times U(1)\big)$ is connected. Furthermore, it is the bundle map from

$$S\big(U(1) \times U(1)\big) \to SU(1+1) \to G_{1,1} = SU(1+1)/S\big(U(1) \times U(1)\big)$$

to

$$U(n) \times U(m) \to U(n+m) \xrightarrow{p} G_{n,m}.$$

Since $\mathrm{Span}_{\mathbb{R}}\{\hat{X}, \widehat{iX}, K\} \perp \mathfrak{u}(m)$, the linearity and the left invariance of vector fields will induce the bundle map from

$$S\big(U(1) \times U(1)\big) \to SU(1+1) \to G_{1,1} = SU(1+1)/S\big(U(1) \times U(1)\big)$$

to

$$U(n) \to U(n+m)/U(m) \xrightarrow{\pi} G_{n,m}$$

thorough the immersed bundle $U(n) \to p^{-1}(S) \to S$ of $U(n) \times U(m) \to U(n+m) \xrightarrow{p} G_{n,m}$. Then the following three different expressions of the bundle equivalences of the Hopf bundles

$$S^1 \to S^3 \to S^2,$$

$$U(1) \to SU(2) \to SU(2)/U(1),$$

and

$$S\big(U(1) \times U(1)\big) \to SU(1+1) \to G_{1,1} = SU(1+1)/S\big(U(1) \times U(1)\big)$$

shows (2). $\square$

By combining Theorems 1.1 and 2.5, we have now

Theorem 2.6. *Let $U(n) \to U(n+m)/U(m) \xrightarrow{\pi} G_{n,m}$ be the natural fibration. Assume the same condition for a complete totally geodesic surface S of Theorem 2.2, and consider the bundle $U(n) \to \pi^{-1}(S) \xrightarrow{\pi} S$. Let γ be a piecewise smooth, simple closed curve on S. Then the holonomy displacement along γ is given by*

$$V(\gamma) = e^{\frac{1}{2}A(\gamma)i} \text{ or } e^{0i} \ \in S^1$$

where $A(\gamma)$ is the area of the region on the surface S surrounded by γ, depending on whether S is a complex submanifold or not.

Remark 2.7. For $n = 1$, we have the following natural bundle $S^1 \to S^{2m+1} \to \mathbb{C}P^m$. Let S be a complete totally geodesic surface in $\mathbb{C}P^m$ and γ be a piecewise smooth, simple closed curve on S. Then the holonomy displacement along γ is given by

$$V(\gamma) = e^{\frac{1}{2}A(\gamma)i} \text{ or } e^{0i} \ \in S^1$$

where $A(\gamma)$ is the area of the region on the surface S surrounded by γ, depending on whether S is a complex submanifold or not. See Corollary 2.4.

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